TUTORIAL NOTES FOR MATH4010

JUNHAO ZHANG

1. COUNTEREXAMPLES

Let us discuss some counterexamples.

1.1. **Banach space.** To construct a Banach space, it is important to give a "good" norm. We show a normed space which is not a Banach space.

Example 1. Let $C([0,1])$ be the space of continuous functions defined on $[0,1]$. Now $C([0,1])$ is endowed with the L^1 norm, i.e.,

$$\|f\|_1 = \int_0^1 |f(x)|dx,$$

for every $f \in C([0,1])$. Then the normed space $C([0,1])$ is not a Banach space.

Proof. We prove $C([0,1])$ is not complete under $\|\cdot\|_1$. Let

$$f_n(x) = \begin{cases} 1, & \frac{1}{2} \le x \le 1, \\ n(x - \frac{1}{2} + \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2}, \\ 0, & 0 \le x \le \frac{1}{2} - \frac{1}{n}, \end{cases}$$

then

$$\|f_n - f_m\|_1 = \frac{|m-n|}{2mn},$$

which implies $\{f_n\}_{n\ge1}$ is a Cauchy sequence. Suppose there exists $f \in C([0,1])$ such that $\{f_n\}_{n\ge1}$ converges to f, we prove that $f \equiv 0$ on $[0, \frac{1}{2})$ and $f \equiv 1$ on $[\frac{1}{2}, 1]$, therefore $f \notin C([0,1])$.

For arbitrary $x_0 \in [0, \frac{1}{2})$, if $f(x_0) \neq 0$, by continuity, there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ sufficiently small such that

$$|f(x)| > \varepsilon_0,$$

for $x \in [x_0 - \delta_0, x_0 + \delta_0]$, therefore

$$\int_{x_0-\delta_0}^{x_0+\delta_0} |f(x)|dx > 2\delta_0\varepsilon_0.$$

However by the convergence of $\{f_n\}_{n\ge1}$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\begin{aligned} \int_0^1 |f(x)|\mathbb{1}_{[x_0-\delta_0,x_0+\delta_0]}dx \le & \int_0^1 |f(x) - f_n(x)|\mathbb{1}_{[x_0-\delta_0,x_0+\delta_0]}dx + \int_0^1 |f_n(x)|\mathbb{1}_{[x_0-\delta_0,x_0+\delta_0]}dx \\ \le & \int_0^1 |f(x) - f_n(x)|dx \\ \le & 2\delta_0\varepsilon_0, \end{aligned}$$

which is a contradiction. Therefore $f \equiv 0$ on $[0, \frac{1}{2})$. Similarly, we can prove $f \equiv 1$ on $[\frac{1}{2}, 1]$. □

Remark 2. It is known that the space $(C([0,1]), \|\cdot\|_\infty)$ is a Banach space. Therefore even for the same set of elements, if the norm is different, the normed space can have an essential difference.

Remark 3. Although $(C([0,1]), \|\cdot\|_1)$ is not complete, it is possible to do the completion of the space to get a Banach space, which is the $L^1(0,1)$ space being given by Lebesgue.

1.2. **Compactness.** Recall that in $\mathbb{R}^n$, we have the following Bolzano-Weierstrass theorem.

Theorem 4 (Bolzano-Weistrass theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

However, by Riesz's lemma, we know that this is not case when we deal with the general normed space or Banach space. Here we give some examples of non-compact subsets.

Example 5. The unit ball

$$B_1 = \{x \in \ell_2 : \|x\|_2 \leq 1\},$$

is closed and bounded, but it is not compact.

Proof. It is clear that B_1 is closed and bounded. Consider $\{e_n\}_{n\geq 1} \subset \ell_2$ defined as

$$e_n(k) = \begin{cases} 0, & \text{if } k \neq n, \\ 1, & \text{if } k = n, \end{cases}$$

therefore for arbitrary $i \neq j$,

$$\|e_i - e_j\|_2 = \sqrt{2},$$

which implies $\{e_n\}_{n\geq 1}$ has no Cauchy subsequences and therefore no convergent subsequences. □

Example 6. The unit ball

$$B_1 = \{f \in C([0,1]) : \|f\|_\infty \leq 1\},$$

is closed and bounded, but it is not compact.

Proof. It is clear that B_1 is closed and bounded. Consider

$$f_n(x) = \begin{cases} 0, & x \geq \frac{1}{n}, \\ 1 - nx, & x \leq \frac{1}{n}, \end{cases}$$

then $\|f_n\|_\infty = 1$ for all $n \geq 1$. Suppose $\{f_n\}_{n\geq 1}$ is compact, then there exists a convergent subsequence $\{f_{n_k}\}_{k\geq 1}$, we denote its limit as f, however

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

therefore $f \notin C([0,1])$ which is a contradiction. □

Email address: jhzhang@math.cuhk.edu.hk